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$(\mathrm{U}(p, q),\ \mathrm{U}(p-1, q))$ is a generalized Gelfand pair

Gerrit van Dijk

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Abstract Denote by $G = \mathrm{U}(p, q)$ the orthogonal group of the sesqui-linear quadratic form $[x,\ y] = x_1\overline{y}_1 + \cdots x_p\overline{y}_p - x_{p+1}\overline{y}_{p+1} - \cdots - x_{p+q}\overline{y}_{p+q}$ on $\mathbb{C}^{p+q}$ and let $H_1 = \mathrm{U}(p-1, q)$ be the stabilizer of the first unit vector e_1. Let $H_0 = \mathrm{U}(1)$ and set $H = H_0 \times H_1$. Define the character χ_l of H by $\chi_l(h) = \chi_l(h_0h_1) = h_0^l$ $(h_0 \in H_0,\ h_1 \in H_1)$ where $l \in \mathbb{Z}$. Define the anti-involution σ on G by $\sigma(g) = \overline{g}^{-1}$. In this note we show that any distribution T on G satisfying $T(h_1gh_2) = \chi_l(h_1h_2)T(g)$ $(g \in G;\ h_1, h_2 \in H)$ is invariant under the anti-involution σ. This result implies that $(G,\ H_1)$ is a generalized Gelfand pair.

Keywords Generalized Gelfand pair · Unitary group · Anti-involution

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1 Introduction

Let $G = \mathrm{U}(p, q)$ be the orthogonal group of the sesqui-linear quadratic form on $\mathbb{C}^n$ $(n = p + q)$ given by

$$[x,\ y] = x_1\overline{y}_1 + \cdots x_p\overline{y}_p - x_{p+1}\overline{y}_{p+1} - \cdots - x_{p+q}\overline{y}_{p+q},$$

and let $H_1 \simeq \mathrm{U}(p-1, q)$ be the stabilizer in G of the first unit vector e_1 of $\mathbb{C}^n$. We are going to show:

Theorem 1 *The pair $(G,\ H_1)$ is a generalized Gelfand pair.*

Let us recall the definition of generalized Gelfand pair [6]. Let $(\pi,\ \mathcal{H})$ be any irreducible unitary representation of G on the Hilbert space $\mathcal{H}$. Then $(G,\ H_1)$ is said to be a generalized Gelfand pair if $\dim \mathcal{H}_{-\infty}^{H_1} \leq 1$. Here $\mathcal{H}_{-\infty}^{H_1}$ is the space of H_1-fixed vectors in $\mathcal{H}_{-\infty}$,

G. van Dijk (✉)
Mathematical Institute, Niels Bohrweg 1, 2333 CA Leiden, The Netherlands
e-mail: dijk@math.leidenuniv.nl

the (continuous) dual of the space $\mathcal{H}_\infty$ of C^∞-vectors. Equivalently one has: any unitary representation of G which can be realized on the space $D'(G/H_1)$ of distributions on G/H_1 decomposes multiplicity free.

In [1] another definition of Gelfand pair is used: for any irreducible admissible smooth Fréchet representation $(\pi,\ E)$ of G one has E^{H_1} is at most one-dimensional. It has been confirmed by the authors that our results imply that the pair $(G,\ H_1)$ is even a Gelfand pair in the sense of [1].

Theorem 1 is certainly true for $p = 1$, the Riemannian case. However in this case a much stronger result holds: any irreducible unitary representation of G decomposes, when restricted to the compact subgroup H_1, multiplicity free. This was proved by Koornwinder in [4] using a criterion which reads as follows: any continuous function F on G satisfying $F(h_1 g h_1^{-1}) = F(g)$ for all $g \in G,\ h_1 \in H_1 \simeq \mathrm{U}(q)$, is invariant under the anti-involution σ of G defined by $\sigma(g) = \overline{g}^{-1}$ $(g \in G)$.

2 A criterion

We shall formulate here a criterion (Theorem 2) which implies Theorem 1.

Let H be the subgroup $H = H_0 \times H_1$ of G where $H_0 = \mathrm{U}(1)$. So any element of H has the form

$$h = \begin{pmatrix} h_0 & 0 \\ 0 & h_1 \end{pmatrix}$$

with $h_0 \in H_0,\ h_1 \in H_1$ Define the character χ_l of H by $\chi_l(h) = h_0^l,\ l \in \mathbb{Z}$.

Let us denote by $d(\theta)$ the diagonal matrix in G with entries $\mathrm{e}^{i\theta}$ $(\theta \in \mathbb{R})$. Clearly $d(\theta)$ belongs to the center of G,

If $(\pi,\ \mathcal{H})$ is an irreducible unitary representation then by Schur's Lemma $\pi(d(\theta))$ acts as a scalar, say $\mathrm{e}^{il\theta}$. Therefore the vectors in $\mathcal{H}_{-\infty}^{H_1}$ transform in the same way under $\pi_{-\infty}(d(\theta))$. This leads to the following criterion.

Theorem 2 *Let $l \in \mathbb{Z}$ be fixed. Any distribution T on G satisfying*

$$T(h_1 g h_2) = \chi_l(h_1 h_2)\, T(g) \quad (g \in G;\ h_1, h_2 \in H)$$

is invariant under the anti-involution of G defined by $\sigma(g) = \overline{g}^{-1}$.

This theorem implies Theorem 1, see [5].

3 Proof of Theorem 2

The idea of the proof was laid down in [2].

Let X_1 be the space defined by

$$X_1 = \{x \in \mathbb{C}^n\ :\ [x,\ x] = 1\}.$$

Then $X_1 \simeq G/H_1$, the isomorphism being given by $p_1 : g \mapsto g \cdot e_1$ $(g \in G)$, where e_1 is the first standard unit vector in $\mathbb{C}^n$.

Let $D(X_1)$ denote the space of complex-valued C^∞-functions on X_1 with compact support. The left G-action on X_1 induces a representation U of G on $D(X_1)$ and by inverse transposition a representation U of G on $D'(X_1)$. We define

$$D'(X_1, l) = \{T \in D'(X_1) \,:\, U_h T = \chi_l(h) T \ (h \in H)\}.$$

This space is naturally isomorphic to the space considered in Theorem 2, the space of distributions T on G that satisfy the (formal) transformation rule

$$T(h_1 g h_2) = \chi_l(h_1 h_2) T(g) \quad (g \in G;\ h_1, h_2 \in H).$$

Let us introduce a map ξ which describes the H_1-orbits on X_1. Let x_1 be the first coordinate of $x \in X_1$. Consider the map $\xi \,:\, X_1 \to \mathbb{C}$ given by $\xi(x) = x_1$. It has the following properties:

- ξ is H_1-invariant,
- ξ is real analytic,
- $\xi(x) = t$ $(t \in \mathbb{C})$ is an H_1-orbit on X_1 if $t\bar{t} \neq 1$,
- ξ has no critical values: rank $d(\xi(x) = 2$ of $t\bar{t} \neq 1$, rank $d\xi(x) = 1$ if $t\bar{t} = 1$.

Moreover we define $Q \,:\, X_1 \to [0,\ \infty)$ by

$$Q(x) = |\xi|^2.$$

We define also the following open subsets of X_1:

$$X_1^0 = \{x \in X_1 \,:\, Q(x) < 1\}$$
$$X_1^1 = \{x \in X_1 \,:\, Q(x) > 0\}.$$

The map Q is left H-invariant, hence both sets are. Therefore we may define for $j = 0, 1$

$$D'(X_1^j, l) = \{T \in D'(X_1^j) \,:\, U_h T = \chi_l(h) T \ (h \in H)\}.$$

Since $X_1^0 \cup X_1^1 = X_1$, the map $T \mapsto (T|_{X_1^0},\ T|_{X_1^1})$ defines a linear bijection from $D'(X_1, l)$ onto the set of pairs $(T_0,\ T_1) \in D'(X_1^0, l) \times D'(X_1^1, l)$ satisfying the matching condition

$$T_0|_{X_1^0 \cap X_1^1} = T_1|_{X_1^0 \cap X_1^1}.$$

We shall study such pairs of distributions. It is sufficient to study the subspaces $D'(X_1^j, l)$ separately, since X_1^j $(j = 0, 1)$ are clearly σ-invariant.

The spaces are treated with different methods. We first deal with the space $D'(X_1^1, l)$ and use the now classical method of Faraut [3] with applying Tengstrand's results. For this purpose we introduce the space $X = X_1/\sim$ where points x and y are identified if $x = \lambda y$ with $\lambda \in \mathbb{C},\ |\lambda| = 1$. Let $x \mapsto \widetilde{x}$ be the natural projection of X_1 onto X. Observe that $X \simeq G/H$. Set $p(g) = \widetilde{p_1(g)}$ $(g \in G)$. Clearly Q is well-defined on X. In addition the point $\widetilde{e}_1$ is an isolated non-degenerate critical point of Q on X. Set $X^1 = \widetilde{X_1^1}$. Notice that the map ξ does not vanish on X_1^1. Therefore one readily sees that multiplication by ξ^l induces a bijection $D'(X^1)^H \to D'(X_1^1, l)$. Here $D'(X^1)^H$ denotes the space of H-invariant distribution on X^1. We have the following results. There is a map $M \,:\, f \mapsto M_f$ which is surjective from $D(X^1)$ onto a space $\mathcal{H}_\eta$, defined by

$$M_f(t) = \int_X f(x)\,\delta(Q(x) - t)\,dx$$

where dx is a G-invariant measure measure on X. One calls $M_f(t)$ the average of f over the surface $Q(x) = t$. The space $\mathcal{H}_\eta$ consists of functions φ on $(0, \infty)$ of the form

$$\varphi(t) = \varphi_0(t) + \eta(t)\varphi_1(t)$$

with $\varphi_0,\ \varphi_1 \in D((0,\infty))$ and η the "singularity function"

$$\eta(t) = Y(1-t)\,(1-t)^{n-2}$$

with Y the Heaviside function: $Y(t) = 1$ of $t \geq 0$, $Y(t) = 0$ if $t < 0$.

Moreover, the transpose M' of M is injective from $\mathcal{H}'_\eta$ to $D'(X^1)$ with image $D'(X^1)^H$. From this we conclude that any bi-H-invariant distribution T on $p^{-1}(X^1) \subset G$ satisfies $T = T^\sigma$. The proof is standard, see [5]. For convenience we repeat the argument here.
Fix Haar measures dg on G and dh on H in such a way that $dg = dx dh$. For $f \in D(G)$ set

$$f^0(x) = \int_H f(gh)\,dh \quad (x = p(g) \in X).$$

Given a bi-H-invariant distribution T on $p^{-1}(X^1) \subset G$ there is a unique H-invariant distribution T_1 on X^1 satisfying $\langle T,\ f\rangle = \langle T_1,\ f^0\rangle$ ($f \in D(p^{-1}(X^1))$), and conversely. Extend the function Q from X to G by $Q(g) = |[g \cdot e_1,\ e_1]|^2$. To show that T is σ-invariant, it is sufficient to show that

$$M_{[(f^\sigma)^0]} = M_{f^0}$$

for all $f \in D(p^{-1}(X^1))$. This is easily checked. For all continuous functions F on $(0,\infty)$ one has

$$\begin{aligned}\int_0^\infty F(t)\,M_{[(f^\sigma)^0]}(t)\,dt &= \int_{X^1} F(Q(x))\,(f^\sigma)^0(x)\,dx\\ &= \int_G F(Q(g))\,f^\sigma(g)\,dg = \int_G F(Q(\sigma(g)))\,f(g)\,dg.\end{aligned}$$

Since $Q(g) = Q(\sigma(g))$ ($g \in G$) we get the result.
Since the function ξ, extended to G by

$$\xi(g) = [g \cdot e_1,\ e_1],$$

is also σ-invariant, we get that any distribution T on $p_1^{-1}(X_1^1)$ satisfying

$$T(h_1 g h_2) = \chi_l(h_1 h_2) T(g) \quad (g \in G;\ h_1, h_2 \in H)$$

is σ-invariant.

We now consider the other space $D'(X_1^0, l)$. Here we use the map ξ. Recall that ξ is a submersion from X_1^0 onto $U = \{z \in \mathbb{C};\ |z| < 1\}$. Its level sets are H_1-orbits on X_1^0. So we have the following lemma.

Lemma 1 *The natural pull-back map*

$$\xi^* : D'(U) \to D'(X_1^0)$$

is injective with image $D'(X_1^0)^{H_1}$.

It is now easy to show, as before, that any distribution T on $p^{-1}(X_1^0) \subset G$ satisfying

$$T(h_1 g h_2) = \chi_l(h_1 h_2) T(g) \quad (g \in G;\ h_1, h_2 \in H)$$

is again σ-invariant, since ξ (extended to G) is.

This completes the proof of Theorem 2.

References

1. Aizenbud, A.A., Gourevitch, D., Sayag, E.: $(O(V \otimes F), O(V))$ is a Gelfand pair for a quadratic space V over a local field F, arXiv: 0711.1471v1 [math.RT] (2007)
2. Charchov, I.: Harmonic Analysis On Line Bundles Over Complex Hyperbolic Spaces. PhD Thesis, Leiden University (2000)
3. Faraut, J.: Distributions sphériques sur les espaces hyperboliques. J. Math. Pures Appl. **58**, 369–444 (1979)
4. Koornwinder, T.H.: A note on the multiplicity free reduction of certain orthogonal and unitary groups. Indag. Math. **44**, 215–218 (1982)
5. van Dijk, G.: On a class of generalized Gelfand pairs. Math. Z. **193**, 581–593 (1986)
6. van Dijk, G., Poel, M.: The Plancherel formula for the pseudo-Riemannian space $SL(n,\mathbb{R})/GL(n-1,\mathbb{R})$. Compos. Math. **58**, 371–397 (1986)